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Cosmology with branes wrapping curved internal manifolds

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Abstract: In this paper we first derive solutions which can be interpreted as branes wrapping nontrivial curved manifolds, and then study their cosmological implications. We find that at early times the branes tend to shrink the internal manifold, while allowing the "unwrapped" dimensions to expand in congruence with what has already been observed in the case when the internal manifold is flat (tori). However, at late times the internal curvature terms become important leading to potentially interesting differences.

Keywords: p-branes, Cosmology of Theories beyond the SM.

Contents

1. Introduction **1**

2. Branes wrapping curved manifolds **3**

3. Brane back reaction **6**

4. Brane gas cosmology **8**

5. Summary and future research **11**

1. Introduction

Inspired by String/M-Theory brane-physics have blossomed in recent years. Branes are p dimensional extended objects embedded in a higher $\widehat{D} \geq p+1$ dimensional universe which arise naturally in String theory as hyperplanes where open strings can end, and in Supergravity/M-Threory as solitons [1]. Mostly, branes have found applications in two virtually opposite set ups: In the "brane-world" scenario [2] the brane dimensions coincide with the three observed dimensions of our universe, while the spatial dimensions perpendicular to the brane, which can be both compact or non-compact correspond to the "unseen internal dimensions". Contrastingly in the "brane-gas" scenario [3, 4] the branes wrap around compact internal dimensions, while the directions perpendicular to the branes become the observed dimensions of our universe. While the brane-world picture have several virtues including being able to address the hierarchy and the cosmological constant problem, the brane gas model can explain why the internal manifold remained small as compared to the observed dimensions, and a simple counting argument also yields the dimensionality of the observed universe to be three! Invoking T-duality brane gas cosmology (BGC) also seems to be able to avoid the big bang singularity. Thus, it may be an interesting venture to combine the two scenarios by considering branes some (three) of whose dimensions are noncompact (which becomes our observed universe) while the others wrap compact internal manifolds, which potentially can be non-trivially curved - the isometries of this curved internal manifold can then be associated with gauge fields living on the brane. In this paper however, we only restrict ourselves to the usual set up of BGC[1] but we generalize the internal manifold from being a flat tori to a curved space.

[1]In the strict sense our study is slightly different from BGC as the dilaton is fixed to be a constant in our case as we will soon find out. However, the analysis is related as the value of the dilaton is supposed to get stabilized at later times any way.

In [4] Brandenberger and Vafa observed that string winding states would tend to prevent expansion since the energy of the states increases if the circumference increases. However, strings with opposite winding numbers can annihilate if their world volume intersects and this happens most efficiently in three dimensions. Thus in a three dimensional subspace the string winding states can annihilate letting these dimensions grow, while strings winding other directions will not be able to annihilate each other efficiently and ultimately fall out of thermal equilibrium thereby stopping the expansion. This stabilization mechanism of the internal manifold by employing 'string gas' was further quantified in [5] and has also been generalized to 'brane gas' scenarios [6] (for more details also see [7]) with essentially similar results, except that now we have a hierarchy in the sizes of the extra dimensions, coming from the contributions of the p-branes, with different p, as in general p-branes annihilate most effectively in $2p+1$ dimensions. Although BGC has been successful in solving several problems associated with standard big bang cosmology, there are some unresolved issues as pointed out in [8]. For example, it is not clear exactly how the three dimensions that we observe today are chosen among all the other dimensions, perhaps through random quantum and thermal fluctuations. In that case it is possible that our universe may contain different patches where different directions have become large! Also, it is known that branes with different values of p can interact with each other, changing their winding numbers, thereby allowing thermal equilibrium to be maintained among the brane winding states. A simple way to eliminate these issues is to assume the universe to be a direct product of three non-compact dimensions and a compact internal manifold. The BGC in such a set up has been extensively studied in [8] when the internal manifold is a tori. Here, we adopt the same approach, but generalize the program to nontrivial curved internal manifolds[2] to study how, if at all, the internal curvature may change the cosmological dynamics. Of course, in this set up the question of why the universe is a product space with specifically three non-compact dimensions is an open issue, and the nice BGC argument concerning the dimensionality of our observed universe becomes redundant. We will come back to this at the end.

For a flat p-brane the p longitudinal (along the brane) directions are flat, or in other words a flat p-brane solution preserves the isometries of the p dimensional flat space. These branes can then wrap around flat p-dimensional tori. If one now considers branes wrapping a curved internal manifold, say M_p then the brane solution should accordingly, preserve the isometries of M_p. In section 2, we derive such p-brane solutions[3] which look like de Sitter black holes from the four dimensional point of view. This is to be expected as after reduction ordinary p-branes resemble black hole solutions, and since our internal background manifold is curved, we are living in an asymptotically de Sitter space-time. We here only obtain the uncharged solution but it may be interesting to study other charged solutions which will have underlying connections to non-abelian black holes (see [11] for a review). In section 3, we obtain the back reaction of these branes on the space-time geometry by analogy with

[2]It should be mentioned that BGC with branes wrapping nontrivial cycles (in Calabi-Yau and K3 spaces) was first studied in [9].

[3]Previously branes wrapping curved dimensions have been studied using boundary state conformal field theory [10], but here we take a supergravity approach.

JHEP02(2004)039

the flat brane solution [8] and check that the stress energy tensor is consistent. As is usual the branes represent delta function sources for the Einstein's equations which can be smoothed out by considering a uniform density of brane gas. In section 4, we study BGC both without and with matter/radiation. We find that the early evolution resembles the picture with flat branes [4, 6, 8]: while the branes wrapping the internal manifold tends to contract them, the internal curvature terms only reinforcing this effect, the noncompact dimensions on the other hand are free to expand as the branes act as pressureless dust. At late times however the picture changes as the curvature term starts dominating over the brane effects. The stabilization of the internal manifold that was achieved in the presence of ordinary matter and branes (the expansion of ordinary matter being balanced by the contraction of the branes) is lost. Preliminary analysis indicates that the internal manifold may also be growing along with the ordinary dimensions. This suggests that we may have to invoke other mechanisms to stabilize the internal manifold at late times, like turning on the fluxes [14]. We conclude by summarizing and commenting on possible future research.

2. Branes wrapping curved manifolds

Supergravity and brane ansatz: we start with a generic bosonic sector[4] of a supergravity (SUGRA) action:

$$\widehat{S} = \frac{1}{16\pi G_{\widehat{D}}}\int dx^{\widehat{D}}\sqrt{-g}\left\{\widehat{R} - \frac{1}{2}\partial_{\widehat{m}}\phi\partial^{\widehat{m}}\phi - \frac{1}{2}\sum_I \frac{1}{n_I!}e^{a_I\phi}F_{n_I}^2 - 2\widehat{\Lambda}\right\}, \qquad (2.1)$$

where ϕ is the dilaton field and field strengths F_{n_I}'s are n_I forms, $I = 1\dots M$ and we have also included a cosmological constant term. Although both in the action and in the field strengths one can have Chern-Simons-like terms it is known that they can be ignored while considering brane like solutions. The field equations that follow from (2.1) are

$$G_{\widehat{m}\widehat{n}} - \frac{1}{2}\left(\partial_{\widehat{m}}\phi\partial_{\widehat{n}}\phi - \frac{1}{2}g_{\widehat{m}\widehat{n}}(\partial\phi)^2\right) -$$

$$-\frac{1}{2}\sum_I \frac{1}{n_I!}e^{a_I\phi}\left(n_I F_{\widehat{m}\widehat{p}_2\dots\widehat{p}_I}F_{\widehat{n}}{}^{\widehat{p}_2\dots\widehat{p}_I} - \frac{1}{2}g_{\widehat{m}\widehat{n}}F_{n_I}^2\right) + \widehat{\Lambda}g_{\widehat{m}\widehat{n}} = 0 \qquad (2.2)$$

$$\frac{1}{\sqrt{-g}}\partial_{\widehat{m}}(\sqrt{-g}g^{\widehat{m}\widehat{n}}\partial_{\widehat{n}}\phi) - \frac{1}{2}\sum_I \frac{1}{n_I!}e^{a_I\phi}F_{n_I}^2 = 0 \qquad (2.3)$$

$$\frac{1}{(n-1)!}\frac{1}{\sqrt{-g}}\partial_{\widehat{m}}(\sqrt{-g}e^{a_I\phi}F^{\widehat{m}\widehat{p}_2\dots\widehat{p}_I}) = 0\,. \qquad (2.4)$$

We now specialize to the uncharged solution, for which we can consistently put the field-strength and the dilaton to zero. We are thus left with only (2.2);[5] (2.3) and (2.4) being trivially satisfied.

Now let us look at the ansatz for our p-brane metric. For simplicity we assume the internal manifold to be a group manifold $G_{\overset{\odot}{D}}$, but one should be able to generalize the results to other homogeneous spaces. Since in our picture the p-brane is wrapping the

[4] In brane solutions all fermionic fields are typically put to zero.

[5] We are essentially in Einstein's gravity with a cosmological constant.

internal manifold, $\overset{\odot}{D}= p$. Moreover, the p-brane metric have to possess the isometries of the internal group manifold. As in the flat p-brane case, this fixes the $\overset{\odot}{D}$ directional components of the metric:

$$g_{\widehat{m}\widehat{n}}(x,y) = \begin{pmatrix} g_{mn}(x) & 0 \\ 0 & \Psi(x) g^K_{\overset{\odot}{m}\overset{\odot}{n}}(y) \end{pmatrix} . \tag{2.5}$$

We employ symbols '$\odot$' and 'no symbol' to denote quantities corresponding to the internal and the external manifold respectively, wherever necessary. Also, x and y denote coordinates charting the observational universe and the internal manifold respectively. Here g^K is the Killing metric[6] possessing the isometries of the group manifold and $\Psi(x)$ is the radion. If we now further impose invariance under time translations and $SO(D)$ rotations among the transverse x directions then we obtain

$$\widehat{ds}^2 = -\widehat{\beta}^2 dt^2 + \widehat{\Phi}^2 dr^2 + \widehat{\Gamma}^2 r^2 d\Omega^2_{D-1} + \Psi^2 \overset{\odot}{d} s^2 \, , \tag{2.6}$$

where all the functions only depend on the radial coordinate r, the brane being located at $r = 0$. Note that the only way this metric differs from the usual flat brane ansatz is that the metric along the brane directions is not flat but corresponds to the Killing metric of the group manifold. At this point one could use (2.6) to solve the field equations (2.2) but it is easier to perform a dimensional reduction which then maps the p-branes to black holes. Since the internal manifold is curved we, in fact, expect the p-branes to look like de Sitter black holes [15].

Reduction and brane solution: to perform the reduction it is convenient to work with the vielbein:

$$\widehat{e}_{\widehat{m}}{}^{\widehat{a}} = \begin{pmatrix} e_m{}^a(x) & 0 \\ 0 & \Psi(x)\, \overset{\odot}{e}_{\overset{\odot}{m}}{}^{\overset{\odot}{a}}(y) \end{pmatrix} ; \qquad \widehat{e}_{\widehat{a}}{}^{\widehat{m}} = \begin{pmatrix} e_a{}^m(x) & 0 \\ 0 & \Psi^{-1}(x)\, \overset{\odot}{e}_{\overset{\odot}{a}}{}^{\overset{\odot}{m}}(y) \end{pmatrix} \tag{2.7}$$

and the "flat-metric"

$$g_{\widehat{a}\widehat{b}} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & g^K_{\overset{\odot}{a}\overset{\odot}{b}} \end{pmatrix} . \tag{2.8}$$

Such a dimensional reduction is known to be consistent and yields

$$\begin{aligned} \widehat{S} &= \frac{1}{16\pi G_{\widehat{D}}} \int dx^{\widehat{D}} \sqrt{-\widehat{g}} (\widehat{R} - 2\widehat{\Lambda}) = \frac{1}{16\pi G_{\widehat{D}}} \int dy^{\overset{\odot}{D}} \sqrt{-\overset{\odot}{g}} \int dx^{D+1} \sqrt{-g}\, \Psi^{\overset{\odot}{D}} [R + K - V] \\ &= \frac{1}{16\pi G_{D+1}} \int dx^{D+1} \sqrt{-g}\, \Psi^{\overset{\odot}{D}} [R + K - V] \, , \end{aligned}$$

where

$$K = -\left(2 \overset{\odot}{D} \frac{\Box \Psi}{\Psi} + \overset{\odot}{D} (\overset{\odot}{D} - 1) \frac{(\partial\Psi)^2}{\Psi^2} \right) ; \qquad V = -\frac{\overset{\odot}{D}}{4} \Psi^{-2} + 2\widehat{\Lambda}$$

[6]One can also employ more general "squashed metric" but we will not consider them in this manuscript.

and

$$\frac{1}{16\pi G_{\widehat{D}}}\int dy^{\overset{\odot}{D}}\sqrt{-\overset{\odot}{g}} = \frac{1}{16\pi G_{\widehat{D}}}V_G = \frac{1}{16\pi G_{D+1}}\,.$$

To obtain the action in the Einstein frame, we perform the well known conformal re-scalings

$$\widehat{e}_{\widehat{a}}{}^{\widehat{m}} = \Delta \widehat{e}'_{\widehat{a}}{}^{\widehat{m}}\,; \qquad \Delta = \Psi'^{\frac{\overset{\odot}{D}}{\widehat{D}-2}}\,. \tag{2.9}$$

Then we have

$$S = \frac{1}{16\pi G_{D+1}}\int dx^{D+1}e'^{-1}\left[R' + K' - V'\right]$$

with

$$K' = -\frac{\overset{\odot}{D}\,(D-1)}{\widehat{D}-2}(\partial'\psi)^2\,; \qquad V' = 2\widehat{\Lambda}e^{-2\frac{\overset{\odot}{D}}{\widehat{D}-2}\psi} - \frac{\overset{\odot}{D}}{4}e^{-2\psi}\,, \tag{2.10}$$

where we have defined

$$\Psi' = e^{\psi}\,.$$

In future we will drop the 'primes'.

The action (2.10) yields field equations for the four dimensional metric and the radion. However, we know that for uncharged p-branes the radion is a constant satisfying

$$\begin{aligned}\frac{\partial V(\psi)}{\partial\psi} &= 0\\ \Rightarrow e^{2\psi} &= \left(\frac{\widehat{D}-2}{8\widehat{\Lambda}}\right)^{\frac{\widehat{D}-2}{D-1}}\,.\end{aligned} \tag{2.11}$$

The potential V then acts as an effective $D+1$ dimensional cosmological constant Λ given by

$$\Lambda \equiv \frac{D-1}{8}e^{-2\psi} = \frac{D-1}{8}\left(\frac{8\widehat{\Lambda}}{\widehat{D}-2}\right)^{\frac{\widehat{D}-2}{D-1}}\,. \tag{2.12}$$

The effective action now reads

$$S = \frac{1}{16\pi G_{D+1}}\int dx^{D+1}e^{-1}\left[R - 2\Lambda\right]. \tag{2.13}$$

De Sitter black hole solutions for (2.13) are known [15]:

$$ds^2 = -\beta^2 dt^2 + \Phi^2 dr^2 + \Gamma^2 r^2 d\Omega^2_{D-1} \tag{2.14}$$

with

$$\Gamma = 1\,; \qquad \beta = \Phi^{-1}\,; \qquad \Phi^2 = \frac{1}{1 - \frac{T_p}{r^{D-2}} - \frac{2\Lambda}{D(D-1)}r^2} \equiv \frac{1}{f(r)}\,. \tag{2.15}$$

The ADM mass of the black hole T_p, arises as an integration constant and is equivalent to the ADM mass or tension of the p-brane [12]. For completeness sake, let us write down the full brane solution

$$\widehat{ds}^2 = \left(\frac{8\widehat{\Lambda}}{\widehat{D}-2}\right)^{\frac{\overset{\odot}{D}}{D-1}}\left(-f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{D-1} + \left(\frac{\widehat{D}-2}{8\widehat{\Lambda}}\right)^{\frac{\widehat{D}-2}{D-1}}\overset{\odot}{d}\,s^2\right). \tag{2.16}$$

JHEP02(2004)039

3. Brane back reaction

Having obtained the brane solution wrapping the extra dimensional manifold, the next task would be to compute its back reaction on the background geometry. The cosmological ansatz for the background metric looks like

$$ds^2 = -\widehat{\beta}^2 dt^2 + \widehat{\alpha}^2 dx^2 + \Psi^2 \overset{\odot}{d} y^2 \,, \tag{3.1}$$

where we take the external three dimensional spatial geometry to be flat. Again it is useful to introduce the vielbeins

$$\widehat{e}_{\widehat{a}}{}^{\widehat{m}} = \begin{pmatrix} e^{-W} & 0 & 0 \\ 0 & e^{-A}\delta_{\underset{a}{\sqcup}}{}^{\overset{\sqcup}{m}} & 0 \\ 0 & 0 & e^{-S}\,\overset{\odot}{e}_{\underset{a}{\odot}}{}^{\overset{\odot}{m}} \end{pmatrix} \tag{3.2}$$

along with the flat metric (2.8). We use the symbol '$\sqcup$' for quantities and indices corresponding to the external spatial dimensions. Here we have also redefined β, α and Ψ as exponentials of W, A and S respectively, which are now to be treated as collective coordinates characterizing the geometry of both the internal and the external manifold, depending only on time. In [8] the stress energy tensor for a flat brane located at x_0 in such a background was already obtained which lends a natural generalization to curved branes:[7]

$$\begin{aligned} T_{00} &= T_p e^{-DA}\delta(x - x_0) \\ T_{\underset{a}{\sqcup}\underset{b}{\sqcup}} &= 0 \\ T_{\underset{a}{\odot}\underset{b}{\odot}} &= -T_p e^{-DA}\delta(x - x_0) g_{\underset{a}{\odot}\underset{b}{\odot}} \,. \end{aligned} \tag{3.3}$$

Consider now the brane gas scenario with say n_l branes located at x_l which implies $\delta(x - x_0) \to \sum_{x_l} n_l \delta(x - x_l)$. As is standard practice we now pass on from a discrete to a continuous distribution of brane gas

$$\sum_{x_l} n_l \delta(x - x_l) \to \int dx' n(x')\delta(x - x') \,,$$

where $n(x)$ is the density of brane gas. Assuming a uniform density then gives us

$$\begin{aligned} T_{00} &= nT_p e^{-DA} \\ T_{\underset{a}{\sqcup}\underset{b}{\sqcup}} &= 0 \\ T_{\underset{a}{\odot}\underset{b}{\odot}} &= -nT_p e^{-DA} g_{\underset{a}{\odot}\underset{b}{\odot}} \,. \end{aligned} \tag{3.4}$$

[7]Although the result physically makes sense because we expect the branes when wrapped around the internal manifold (curved or flat) to behave as delta function sources in the transverse (observable) directions, it would be nice to derive it ab initio starting from, for example, the Polyakov brane action. This however would require a better understanding of the brane dynamics which may also lead to some corrections to (3.3).

JHEP02(2004)039

This stress-energy tensor differs from the usual brane gas stress-energy tensor (see for example [6]) which has an equation of state parameter $-p/(\widehat{D}-1)$ along all dimensions. As we will see later, (3.4) corresponds to equation of state parameters -1 along the wrapped compact dimensions and 0 for the non-compact unwrapped directions. This apparent discrepancy arises because we do not perform an average of brane wrappings over all dimensions as in our case the wrapped dimensions are fixed and topologically distinct from the non-compact dimensions. In the conventional BGC one has to perform the average as all dimensions are equivalent and then the averaged equation of state parameter will indeed be given by

$$\frac{(-1).\overset{\odot}{D}+0.D}{\widehat{D}-1} = -\frac{\overset{\odot}{D}}{\widehat{D}-1} \, .$$

Before proceeding any further, let us check that the stress energy tensor is indeed consistent, i.e. it satisfies the divergence free condition

$$\nabla_{\widehat{a}} T^{\widehat{a}\widehat{b}} = 0 \, . \tag{3.5}$$

Now

$$\nabla_{\widehat{a}} T^{\widehat{a}\widehat{b}} = e_{\widehat{a}} T^{\widehat{a}\widehat{b}} + \omega^{\widehat{a}}{}_{\widehat{c}\widehat{a}} T^{\widehat{c}\widehat{b}} + \omega^{\widehat{b}}{}_{\widehat{c}\widehat{a}} T^{\widehat{a}\widehat{c}}$$

and the non-zero connection co-efficients for (3.2) are given by

$$\begin{aligned}
\widehat{\omega}^{\overset{\sqcup}{a}}{}_{0\overset{\sqcup}{c}} &= \delta_{\overset{\sqcup}{c}}{}^{\overset{\sqcup}{a}} e^{-W} \dot{A} \\
\widehat{\omega}^{\overset{\odot}{a}}{}_{0\overset{\odot}{c}} &= \delta_{\overset{\odot}{c}}{}^{\overset{\odot}{a}} e^{-W} \dot{S} \\
\widehat{\omega}^{\overset{\odot}{a}}{}_{\overset{\odot}{b}\overset{\odot}{c}} &= e^{-S} \omega^{\overset{\odot}{a}}{}_{\overset{\odot}{b}\overset{\odot}{c}}
\end{aligned} \tag{3.6}$$

Then trivially

$$\begin{aligned}
\nabla_{\widehat{a}} T^{\widehat{a}\overset{\sqcup}{b}} &= 0 \\
\nabla_{\widehat{a}} T^{\widehat{a}0} &= e_{\widehat{a}} T^{\widehat{a}0} + \omega^{\widehat{a}}{}_{\widehat{c}\widehat{a}} T^{\widehat{c}0} + \omega^{0}{}_{\widehat{c}\widehat{a}} T^{\widehat{a}\widehat{c}} \\
&= e^{-W} \partial_t (n T_p e^{-DA}) + e^{-W} (D\dot{A} + \overset{\odot}{D}\, \dot{S}) n T_p e^{-DA} - e^{-W} n T_p \overset{\odot}{D}\, \dot{S} e^{-DA} = 0 \, .
\end{aligned} \tag{3.7}$$

Finally

$$\begin{aligned}
\nabla_{\widehat{a}} T^{\widehat{a}\overset{\odot}{b}} &= e_{\widehat{a}} T^{\widehat{a}\overset{\odot}{b}} + \omega^{\widehat{a}}{}_{\widehat{c}\widehat{a}} T^{\widehat{c}\overset{\odot}{b}} + \omega^{\overset{\odot}{b}}{}_{\widehat{c}\widehat{a}} T^{\widehat{a}\widehat{c}} \\
&= 0 + \omega^{\overset{\odot}{a}}{}_{\overset{\odot}{d}\overset{\odot}{a}} T^{\overset{\odot}{d}\overset{\odot}{b}} + \omega^{\overset{\odot}{b}}{}_{\overset{\odot}{d}\overset{\odot}{a}} T^{\overset{\odot}{a}\overset{\odot}{d}} = 0
\end{aligned}$$

since $\omega^{\overset{\odot}{a}}{}_{\overset{\odot}{d}\overset{\odot}{a}} = 0$ for unimodular groups, and compact groups are unimodular.

4. Brane gas cosmology

Without matter: having obtained the energy momentum tensor for the brane gas, we can now proceed to obtain Einstein's equations of motion. In order to do that we first compute the Einstein tensor for the geometry (3.2), (2.8).

$$\begin{aligned}
G_{00} &= e^{-2W}\left[\frac{1}{2}D(D-1)\dot{A}^2+\frac{1}{2}\overset{\odot}{D}(\overset{\odot}{D}-1)\dot{S}^2+D\overset{\odot}{D}\dot{A}\dot{S}+\frac{1}{2}\overset{\odot}{D}\overset{\odot}{\lambda}e^{2(W-S)}\right]\\
G_{\underset{a}{\sqcup}\underset{b}{\sqcup}} &= -\delta_{\underset{a}{\sqcup}\underset{b}{\sqcup}}e^{-2W}\Bigg[(D-1)\ddot{A}+\overset{\odot}{D}\ddot{S}+\frac{1}{2}D(D-1)\dot{A}^2+\frac{1}{2}\overset{\odot}{D}(\overset{\odot}{D}+1)\dot{S}^2+\\
&\qquad +\overset{\odot}{D}(D-1)\dot{A}\dot{S}-(D-1)\dot{A}\dot{W}-\overset{\odot}{D}\dot{S}\dot{W}+\frac{1}{2}\overset{\odot}{D}\overset{\odot}{\lambda}e^{2(W-S)}\Bigg]\\
G_{\underset{a}{\odot}\underset{b}{\odot}} &= -g_{\underset{a}{\odot}\underset{b}{\odot}}e^{-2W}\Bigg[D\ddot{A}+(\overset{\odot}{D}-1)\ddot{S}+\frac{1}{2}D(D+1)\dot{A}^2+\frac{1}{2}\overset{\odot}{D}(\overset{\odot}{D}-1)\dot{S}^2+\\
&\qquad +D(\overset{\odot}{D}-1)\dot{A}\dot{S}-D\dot{A}\dot{W}-(\overset{\odot}{D}-1)\dot{S}\dot{W}+\frac{1}{2}(\overset{\odot}{D}-2)\overset{\odot}{\lambda}e^{2(W-S)}\Bigg]. \qquad (4.1)
\end{aligned}$$

Einstein's field equations then read

$$G_{\widehat{a}\widehat{b}}=\kappa^2T_{\widehat{a}\widehat{b}}\,. \qquad (4.2)$$

To analyse these equations in more detail it is convenient to choose the 'canonical gauge'[8] [8]:

$$W=DA+\overset{\odot}{D}S \qquad (4.3)$$

After some recombinations one obtains

$$\begin{aligned}
(\widehat{D}-2)\ddot{A} &= (\overset{\odot}{D}+1)\kappa^2nT_pe^{2\overset{\odot}{D}S+DA}\\
(\widehat{D}-2)\ddot{S} &= -(D-2)\kappa^2nT_pe^{2\overset{\odot}{D}S+DA}-\overset{\odot}{\lambda}(\widehat{D}-2)e^{2(\overset{\odot}{D}-1)S+2DA}. \qquad (4.4)
\end{aligned}$$

The third equation is not expected to be linearly independent as usually happens in general relativity, though it can constraint the initial conditions. These equations look the same as with branes wrapping flat tori [8], the only addition being the internal curvature term. However, the curvature term has the same sign as that of the brane term and thus it can only enhance the brane gas effect of trying to contract the extra dimensions. Unfortunately we couldn't find an analytical solution as it was possible in the flat case, but numerical evolution of the equations exemplify the behaviour.

As expected we find that while the unwrapped dimensions grow, the wrapped dimensions shrink (see figure 1) thus providing an explanation for the differential size of the external and the internal dimensions.

[8]This is the gauge where in the action the kinetic terms for the variables S and A are canonical, and hence we expect Newton's law type equations which provide a clear picture of the dynamics. One can always go back to the more standard $W=0$ gauge.

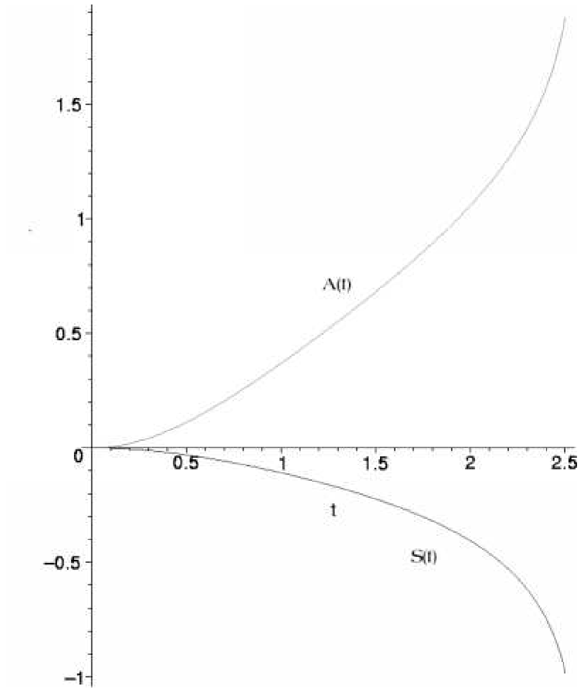

Figure 1: Typical evolution of $A(t)$ and $S(t)$ in planckian units.

With matter: let us now make the picture more realistic by adding matter/radiation:

$$
\begin{aligned}
T^M_{00} &= \rho \\
T^M_{\underset{a}{\sqcup}\underset{b}{\sqcup}} &= \delta_{\underset{a}{\sqcup}\underset{b}{\sqcup}} p \\
T^M_{\underset{a}{\odot}\underset{b}{\odot}} &= g_{\underset{a}{\odot}\underset{b}{\odot}} \overset{\odot}{p}\ .
\end{aligned}
\tag{4.5}
$$

One can solve for the energy density ρ using the equations of state

$$
\begin{aligned}
p &= \omega\rho \\
\overset{\odot}{p} &= \overset{\odot}{\omega}\ \rho
\end{aligned}
\tag{4.6}
$$

and the divergence equation

$$\nabla_{\widehat{a}} T^{\widehat{a}0} = 0\,.$$

One obtains exactly the same result as for the flat case:

$$\rho = \rho_0 e^{-D(1+\omega)A - \overset{\odot}{D}(1+\overset{\odot}{\omega})S}\,. \tag{4.7}$$

It is easy to check that other equations in (3.5) are automatically satisfied. It is interesting to note that the brane stress energy tensor is also of the same form as (4.5) and (4.6) with $\omega = 0$, $\overset{\odot}{\omega} = -1$ and $\rho_0 = nT_p$.

The addition of radiation/matter modifies the equations of motion to

$$(\widehat{D}-2)\ddot{A} = (\overset{\odot}{D}+1)\kappa^2 n T_p e^{2\overset{\odot}{D}S+DA} + \kappa^2\rho_0(1+(\overset{\odot}{D}-1)\omega - \overset{\odot}{D}\overset{\odot}{\omega})e^{D(1-\omega)A+\overset{\odot}{D}(1-\overset{\odot}{\omega})S}$$

$$
\begin{aligned}
(\widehat{D}-2)\ddot{S} = &-(D-2)\kappa^2 n T_p e^{2\overset{\odot}{D}S+DA} - \overset{\odot}{\lambda}(\widehat{D}-2)e^{2(\overset{\odot}{D}-1)S+2DA} + \\
&+\kappa^2\rho_0(1+(D-1)\overset{\odot}{\omega}-D\omega)e^{D(1-\omega)A+\overset{\odot}{D}(1-\overset{\odot}{\omega})S}\,.
\end{aligned}
\tag{4.8}
$$

It is clear that the evolution would be complex and there could be several regimes depending upon the specific values of the equation of state parameters and the values of S and A. In general one cannot solve these equations analytically and to comprehensively understand the dynamics one will have to resort to numerical simulations. However, let us consider a likely scenario:

It is desirable to account for an inflationary phase in the beginning. Thus one can assume that initially ρ_0 comes from a cosmological term (or perhaps an effective cosmological term arising from slow rolling of a scalar field, with $\rho_0 = \rho_{\texttt{inflaton}}$) with $\omega = \overset{\odot}{\omega} = -1$. It is easy to see from the evolution equations that when both A and S are small enough then the dominant exponents are associated with the internal curvature and the brane contributions and hence the evolution would proceed similar to the case when no matter was present. A will expand while S will shrink giving us the origin of the differential size of A and S. As A increases, there will come a point when cosmological term starts to dominate over the brane term. Also note, the internal curvature term always dominate over the brane term because S is small while A is large. Thus the phase of differential 'growth' will soon give way to an inflationary phase where the size of the internal manifold remains approximately constant while that of the external manifold increases exponentially (this is nothing but the familiar de Sitter vacuum). After the end of inflation, presumably the universe becomes radiation and later matter dominated:

$$\rho_{\texttt{inflaton}} \rightarrow \rho_{\texttt{radiation}} \rightarrow \rho_{\texttt{matter}}\,.$$

At this point one may wonder as to what kind of equation of state parameter does matter and radiation obey along the extra dimensions? Although the answer is not clear it seems natural that since the internal manifold is now very small the wave-function of the matter/radiation particles will wrap around the extra-dimensional manifold and thus effectively behave in much the same way as branes do, or in other words have $\overset{\odot}{\omega} = -1$. Since by this time we expect most of the branes to have annihilated the evolution would be governed by a competition between the internal curvature and the radiation/matter terms, unlike between brane and radiation/matter as was discussed in [8] for the flat case.

One can now try to find exact solutions (assuming $T_p = 0$) of (4.8) by matching the exponents [13, 8]. We find it convenient to directly work in the $W = 0$ gauge for this purpose, where the independent equations look like

$$\frac{1}{2}D(D-1)\dot{A}^2 + \frac{1}{2}\overset{\odot}{D}(\overset{\odot}{D}-1)\dot{S}^2 + D\overset{\odot}{D}\dot{A}\dot{S} + \frac{1}{2}\overset{\odot}{D}\overset{\odot}{\lambda}e^{-2S} = \kappa^2\rho_0 e^{-D(1+\omega)A-\overset{\odot}{D}(1+\overset{\odot}{\omega})S}$$

and

$$
\begin{aligned}
D\ddot{A} + (\overset{\odot}{D}-1)\ddot{S} + \frac{1}{2}D(D+1)\dot{A}^2 + \frac{1}{2}\overset{\odot}{D}(\overset{\odot}{D}-1)\dot{S}^2 + \\
+D(\overset{\odot}{D}-1)\dot{A}\dot{S} + \frac{1}{2}(\overset{\odot}{D}-2)\overset{\odot}{\lambda}e^{-2S} = -\kappa^2\overset{\odot}{\omega}\rho_0 e^{-D(1+\omega)A-\overset{\odot}{D}(1+\overset{\odot}{\omega})S}\,.
\end{aligned}
\tag{4.9}
$$

JHEP02(2004)039

To match the exponents we make the following ansatz

$$A(t) = ln(a_0) + a_1 ln(t)\,; \qquad S(t) = ln(s_0) + s_1 ln(t)\,. \tag{4.10}$$

With (4.10) since $\dot{A}, \dot{S}, \ddot{A}, \ddot{S}$, all go as $\sim t^{-2}$ we impose

$$-D(1+\omega)a_1 - \overset{\odot}{D}(1+\overset{\odot}{\omega})s_1 = -2$$

and

$$\begin{aligned} -2s_1 &= -2 \\ \Rightarrow s_1 &= 1 \text{ and } a_1 = \frac{2}{3(1+\omega)} \end{aligned} \tag{4.11}$$

for $D = 3$ and $\overset{\odot}{\omega} = -1$. For both radiation ($\omega = 1/3$) and matter ($\omega = 0$) this gives the usual scaling laws for the size of the external universe, $e^A \sim t^{1/2}$ and $e^A \sim t^{2/3}$ respectively. Matching the exponents however still leaves us to satisfy two equations with two unknowns (a_0, s_0). In the flat case with branes and matter/radiation, one could indeed find such a solution set, but unfortunately it fails in our case (although solutions exist for $\omega \to 0^-$). However, as remarked in [8] (4.11) can still be taken to indicate a general behaviour of the solutions and we here take a similar view point. We also observe that unlike in the flat case where an interplay between the brane contraction and expansion due to pressureless dust could stabilize the internal manifold, the internal manifold here grows linearly with time! This is because the interplay of curvature terms and matter gives us a "potential hill" rather than a valley. To exemplify this consider the evolution of S in (4.8) with $T_p = 0$ and in the presence of a pure cosmological constant[9]. In the equation of motion of S the A dependence factorizes and the evolution is governed by an effective potential of the form

$$V_{\texttt{eff}}(S) \sim C_{\texttt{curv}} e^{2(\overset{\odot}{D}-1)S} - C_{\texttt{cos/mat}} e^{2\overset{\odot}{D}S}\,. \tag{4.12}$$

Clearly there is an unstable potential hill and because of the cosmological/matter term S tends to grow once to the right hand side of the hill. This suggests that one perhaps needs to incorporate other forms of stabilization mechanism, for example by turning on the flux fields [14], to be able to stabilize the internal manifold when it is curved.

5. Summary and future research

In this paper we have tried to describe a framework to study BGC with branes wrapping curved group manifolds rather than flat tori which has been extensively studied. We find that although at earlier times the dynamics is similar to the flat case, the branes now reinforced by the internal curvature effects again tries to contract the wrapped compact dimensions, the later dynamics is different as the internal curvature terms become important. In particular, there seems to be a regime when the competition between the internal

[9]Note that in our approach the equation of state for matter along the extra dimensions is the same as that of a pure cosmological constant term.

curvature terms and the usual matter-radiation contributions allow the external universe to scale in the usual way $\sim t^{1/2}$ for radiation and $\sim t^{2/3}$ for matter at least approximately. However the analysis carried out here is only qualitative and to have a better understanding of the dynamics one needs to perform a thorough numerical calculation.

One can also try to explore whether one can generate inflation[16] in these models without any explicit $\rho_{\mathtt{infl}}$. Our preliminary numerical analysis does seem to suggest (see figure 1) that the external universe undergoes an accelerated phase of expansion at early times while the internal manifold is contracting. Such an inflationary regime will come to an end when branes annihilate to produce reheating. A related issue would be to study the stability of the branes in the first place. This may reveal that a brane gas cannot be supported for some of the group manifolds and thus their smallness today cannot be explained in the BGC framework.

Finally, it would be interesting to try and compute the annihilation rates of branes wrapping different subgroups (or coset spaces) of the internal group manifold and then apply the same reasoning as was done for the tori to see if one can generate a hierarchy of size (or squashing) within the full group manifold. It would be even more interesting to be able to start out with a non-compact simple group manifold and carry out this analysis to see whether the non-compact directions really become large while the compact ones are constrained by branes. This would then at least be able to explain why today we observe a product structure of the external and the internal universe. Of course this still will not explain the dimensionality of the observed universe.

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JHEP02(2004)039